Integration radius as a parameter separating convergent and divergent spherical harmonic series of topography-implied gravity

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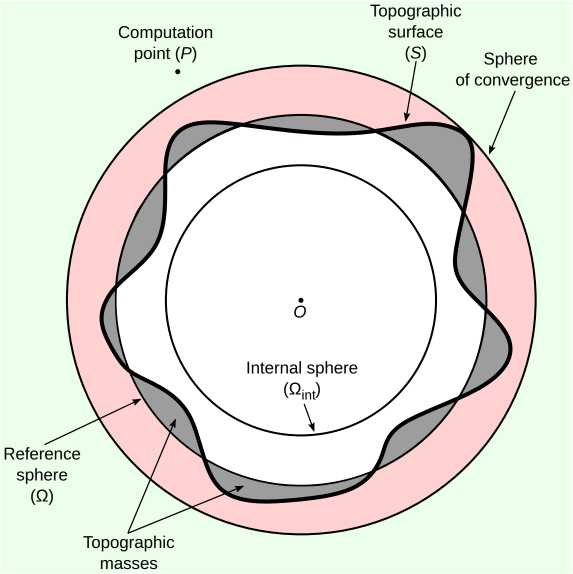
Motivation

Global spectral gravity forward modelling



Global forward modelling:

$$\bar{H}_{nm} \longrightarrow \bar{V}_{nm}$$

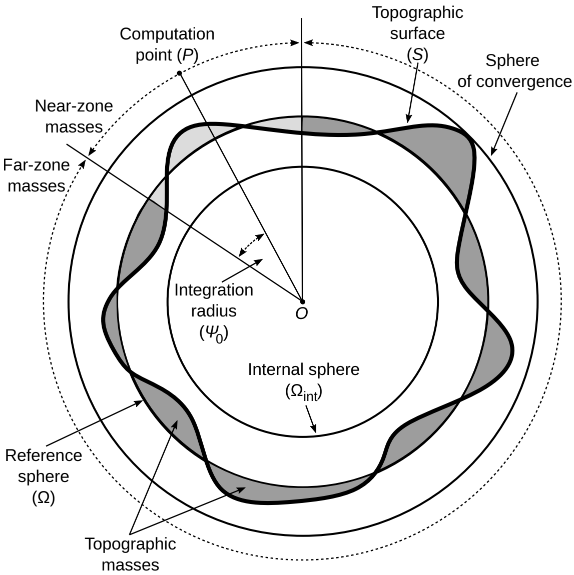
Potential series:

$$V = \frac{GM}{R} \sum_{n=0}^{\infty} \left(\frac{R}{r} \right)^{n+1} \times \sum_{m=-n}^{n} \bar{V}_{nm} \, \bar{Y}_{nm}(\varphi, \lambda)$$

Series behaviour:

- Convergence guaranteed
- Convergence not guaranteed

Cap-modified spectral gravity forward modelling



Cap-modified forward modelling:

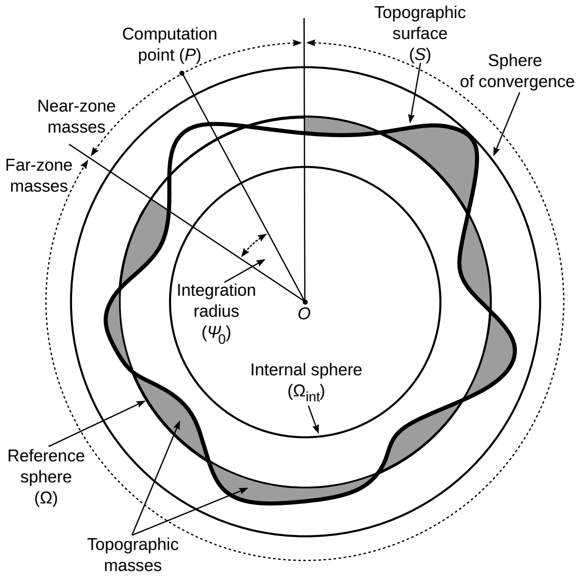
$$\bar{H}_{nm} \longrightarrow \bar{V}^{j}_{nm}(r, \psi_0)$$

with $j =$ 'N' (near-zone) or 'F' (far-zone).

Potential series:

$$V^{j} = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{V}^{j}_{nm}(r, \psi_0) \times \bar{Y}_{nm}(\varphi, \lambda)$$

Far-zone effects



Cap-modified forward modelling:

$$\bar{H}_{nm} \longrightarrow \bar{V}^{\mathrm{F}}_{nm}(r, \psi_0)$$

Potential series:

$$V^{\mathrm{F}} = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{V}^{\mathrm{F}}_{nm}(r, \psi_0) \times \bar{Y}_{nm}(\varphi, \lambda)$$

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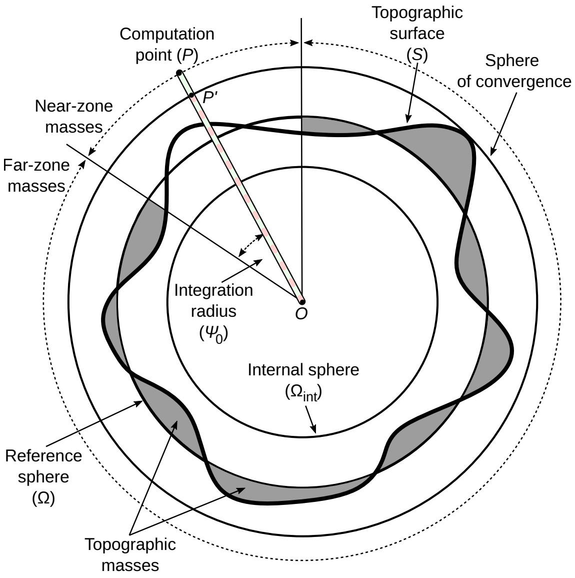

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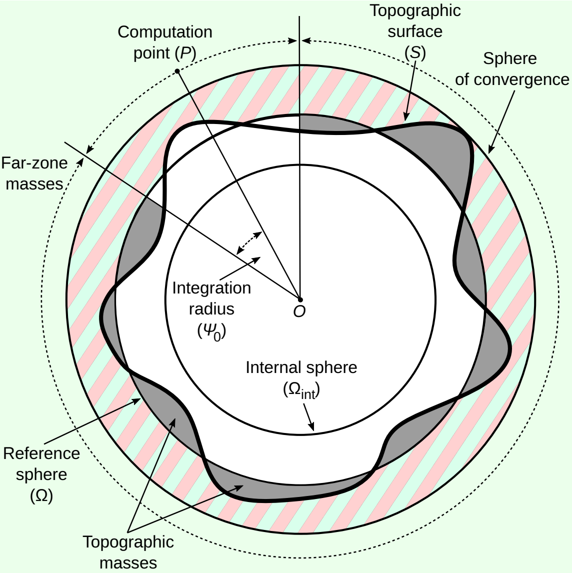
Convergence on $\overline{PP'}$.

But what about $\overline{P'O}$?

Far-zone effects



What is the convergence region for far-zone effects?

Theory

Formulation of the problem

- For a constant mass density and $r_{\mathrm{S}} > R_{\mathrm{int}}$, Newton's integral reads:

$$V^{\mathrm{F}}(r, \varphi, \lambda) = G\, \rho \int\limits_{\psi=\psi_0}^{\pi} \int\limits_{\alpha=0}^{2\pi} \int\limits_{r'=R_{\mathrm{int}}}^{r_{\mathrm{S}}} \frac{(r')^2}{\ell(r, \psi, r')} \mathrm{d}r' \, \mathrm{d}\alpha \sin\psi \, \mathrm{d}\psi. \tag{1}$$

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- Taylor expansion of Newton's kernel [Martinec(1998)]:

$$\frac{(r')^2}{\ell(r, \psi, r')} = R_{\mathrm{int}}^2 \sum_{i=0}^{\infty} \frac{1}{i!} M_i(r, \psi, R_{\mathrm{int}}) \left(\frac{r' - R_{\mathrm{int}}}{R_{\mathrm{int}}} \right)^i. \tag{2}$$

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- Substituting Eq. (2) into (1), interchanging the order of the summation and integrations and after some derivations, we get:

$$V^{\mathrm{F}}(r,\varphi,\lambda) = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{V}_{nm}^{\mathrm{F}}(r,\psi_0)\, \bar{Y}_{nm}(\varphi,\lambda). \qquad (3)$$

Formulation of the problem

- The summation and integrations can be interchanged only if the Taylor series (2) converges uniformly. [Martinec(1998)] has shown that the radius of convergence of (2) for $r = R_{\mathrm{int}}$ is ℓ. This implies

$$\psi_0 > 2 \arcsin \left(\frac{\max(\hat{H}_{\mathrm{int}}(\varphi', \lambda'))}{2\, R_{\mathrm{int}}} \right) \approx \frac{\max \left(\hat{H}_{\mathrm{int}}(\varphi', \lambda') \right)}{R_{\mathrm{int}}}. \tag{4}$$

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- The radius of convergence for $r > R_{\mathrm{int}}$ is not know.
- **Hypothesis**: The radius of convergence of the Taylor series (2) for $r \geq R_{\mathrm{int}}$ is ℓ.
  This implies

$$\psi_0 > \operatorname{arccos} \left(\frac{r^2 + R_{\mathrm{int}}^2 - (\max(\hat{H}_{\mathrm{int}}))^2}{2\, r\, R_{\mathrm{int}}} \right). \qquad (5)$$

- If (5) is satisfied and **<u>if</u>** the hypothesis is true, then we have a proof that external spherical harmonic expansions of far-zone gravitational effects converge *even on the topography and below it down to the R_{int}-sphere.*

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- The hypothesis has not yet been proven. The main obstacle is the form of the kernels M_i,

$$
\begin{aligned}
M_i(r, \psi, R_{\mathrm{int}}) = & \frac{1}{\ell} \sum_{s=1}^{i-1} \frac{i!\,(i-2)!}{(i-s-1)!\,(s-1)!} \left(\frac{r}{\ell}\right)^{i+1-s} \\
& \times \sum_{t=0}^{i+1-s} (-1)^{\frac{1}{2}(3i+1-s+t)} \frac{(i+2-s-t)!!\,(i-s+t)!!}{(i+2-s-t)!\,t!} \\
& \times \left(\frac{z}{\ell}\right)^{t} .
\end{aligned}
\tag{6}
$$

Numerical experiments

Radius of convergence of the TS: Numerical approach

Using the root test $C = \limsup_{i \to \infty} \sqrt[i]{|c_i|}$, the radius of convergence is $D = \frac{1}{C}$.

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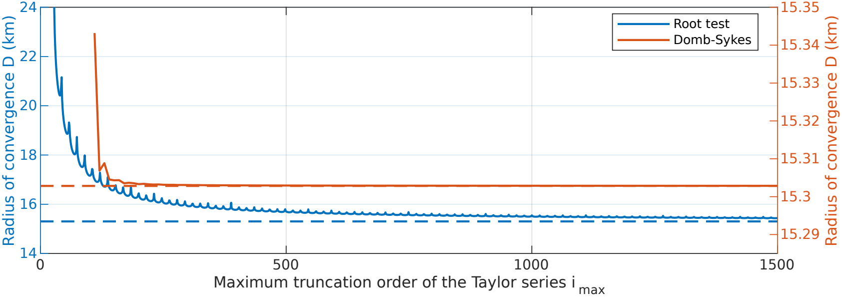

Radius of convergence of the Taylor series (2) for $r = R_{\text{int}} + 15{,}000$ m, $\psi_0 = 0.1^\circ$ and $R_{\text{int}} = 1{,}728{,}200$ m. Both methods have their own vertical axis, given that the Domb–Sykes method converges significantly faster. For each axis, shown is also the (same) prediction of the radius of convergence ℓ (dashed lines).

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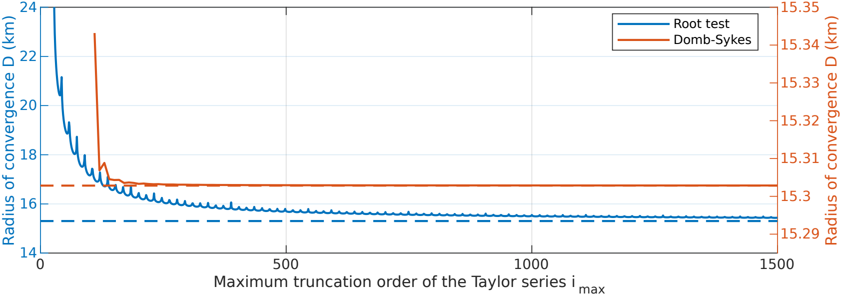
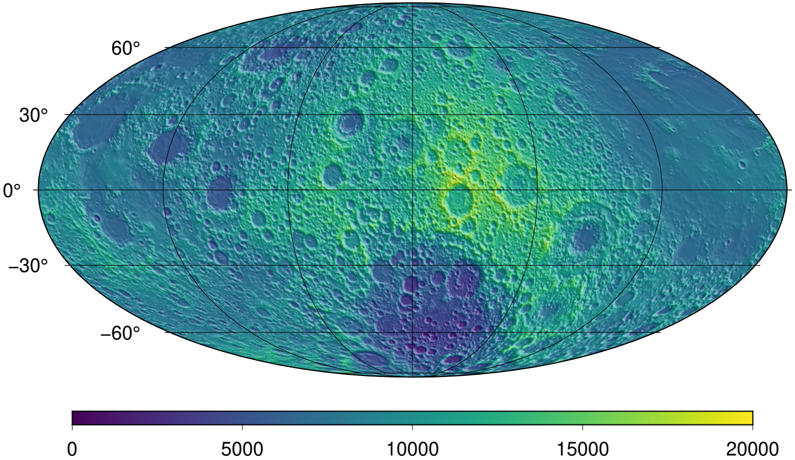


Radius of convergence of the Taylor series (2) for $r = R_{\mathrm{int}} + 15{,}000\ \mathrm{m}$, $\psi_0 = 0.1^\circ$ *and* $R_{\mathrm{int}} = 1{,}728{,}200\ \mathrm{m}$. *Both methods have their own vertical axis, given that the Domb–Sykes method converges significantly faster. For each axis, shown is also the (same) prediction of the radius of convergence* ℓ *(dashed lines).*

The numerical experiments do not invalidate the hypothesis.
This is far from a formal proof, though.

Design of the experiment

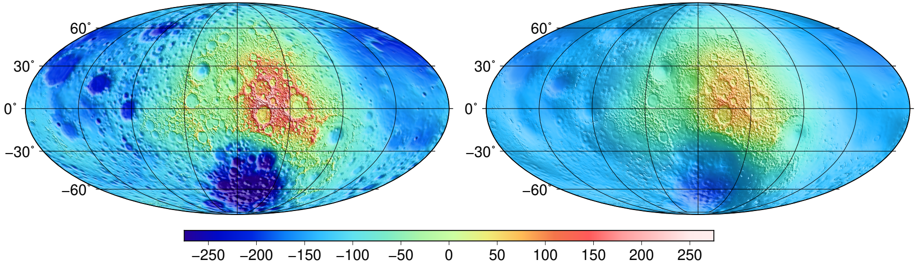
Gravitating body: Moon's topographic masses up to degree 2160.



Moon's topographic heights (m) referenced to the sphere of radius $R_{\text{int}} = 1{,}728{,}200\ \text{m}$.

Reference gravity disturbances: Spatial-domain forward modelling for several ψ_0, the threshold being $\approx 0.663°$ (≈ 20.114 km).

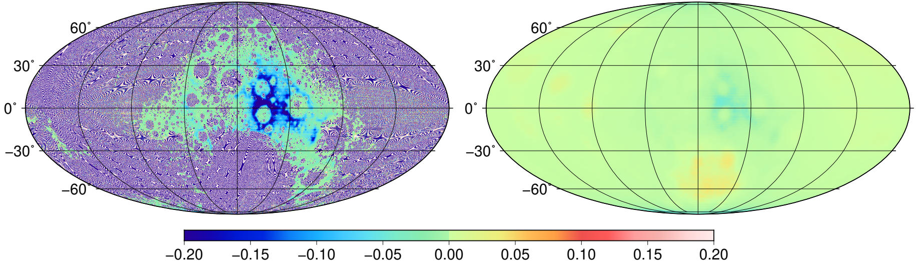
ℓ_0 (km)	2.5	5.0	7.5	10.0	12.5	15.0	20.0	30.0	50.0	100.0
$\psi_0 = \ell_0/R$ (deg)	0.08	0.16	0.25	0.33	0.41	0.49	0.659	0.82	1.65	3.30
Series behaviour	D	D	D	D	D	D	D	C	C	C



Reference far-zone gravity disturbances (mGal) from a divergence-free spatial-domain Newtonian integration residing on the surface of the field-generating masses. Left: $\ell_0 = 10$ km, right $\ell_0 = 50$ km.

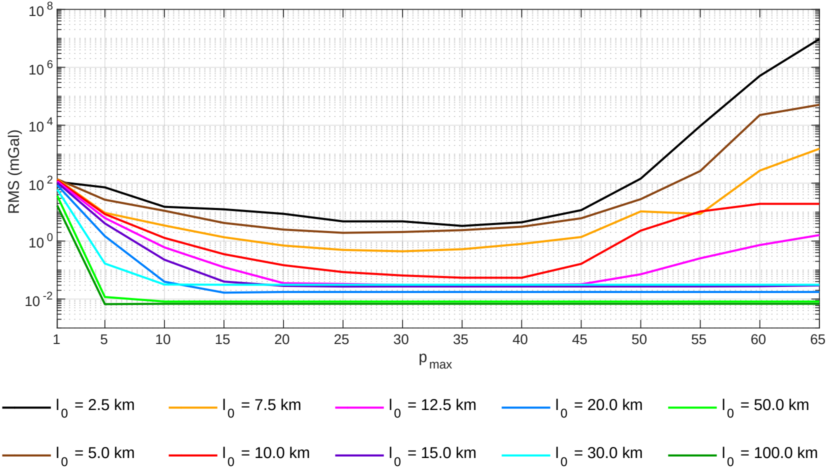
Results

- Maximum harmonic degree of the potential series: 10,800
- Number of topography powers: 65



Differences between spectral- and spatial-domain far-zone gravity disturbances (mGal) on the surface of the field-generating masses. Left: $\ell_0 = 10\ \mathrm{km}$*, right* $\ell_0 = 50\ \mathrm{km}$*. The discrepancies on the left reach thousands of mGal.*

Results



Differences between spectral- and spatial-domain far-zone gravity disturbances on the surface of the field-generating masses for various integration radii ℓ_0 as a function of the maximum topography power $p_{\max}$. For $p_{\max} = 1$, the maximum harmonic degree is 2160 and 10,800 otherwise.

Conclusions

Conclusions

- Insights into convergence/divergence of spherical harmonics
- We formulated a hypothesis that separates between convergent and divergent far-zone spherical harmonic series
- Numerical experiments do not invalidate the hypothesis
- Rigorous proof still missing
- Applications in full-scale high-resolution gravity forward modelling:
  - Near-zone effects with spatial methods
  - Far-zone effects with spectral methods

References

Z. Martinec.
Boundary-Value Problems for Gravimetric Determination of a Precise Geoid.
Springer-Verlag, Berlin, Heidelberg, 1998.

Thank you for your attention!

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Fast high-degree SHA/SHS library in C/Python: www.charmlib.org